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STELLAR TURBULENT CONVECTION. I. THEORY

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ABSTRACT

The description of stellar turbulent convection requires a minimum of five coupled, time-dependent, nonlocal, differential equations for the five variables: turbulent kinetic energy, turbulent potential energy, turbulent pressure, convective flux, and energy dissipation. Any fewer number of equations makes the model local. In this paper, we present the following results:

1. We derive the five coupled equations using a new turbulence model. The physical foundations and the turbulence statistics on which the model was tested are discussed. The model is able to reproduce the high Rayleigh number laboratory and direct numerical simulation data corresponding to medium-to-high values of the Peclet number (a measure of the efficiency of convection).
2. One of the major difficulties for any stellar convective model is the description of the low-efficiency, low Pe number region in which the physical timescale is no longer the turbulent timescale but the radiative one. No previous turbulence model has been able to incorporate these multiple timescales within the same framework properly. The present model does.
3. Overshooting is an unsolved problem in stellar structure. Its solution requires not only the above ingredients, but an additional one, a nonlocal model. This is because in the stably stratified region where $\nabla - \nabla_{\rm ad} < 0$, the only source of energy is diffusion, a nonlocal process. We discuss why the expressions used thus far to describe diffusion terms are inadequate. We then present a model that was successfully tested against LES data on the convective planetary boundary layer.
4. We analyze the nonlocal models of Gough and Xiong and discuss the approximations that are required to derive them from the full set of equations.
5. We discuss a model that relates the up/down drafts filling factors found by DNS/LES to the skewness of the velocity field which can be computed from the turbulence model. The results from DNS/LES and this model can thus be cross-checked.
6. We show that the stationary, local limit of the model reproduces recent local models (independently derived) which have been successfully tested against a variety of astrophysical data.
7. We discuss the fact that *if* the dissipation ϵ is described by a local model with a mixing length l (as done by all authors thus far), the remaining nonlocal equations exhibit divergences which preclude a physical solution to be found. OV results based on this method may be a coincidence since they are arrived at by fine tuning a coefficient.
8. The role of compressibility is discussed.

Subject headings: convection — diffusion — stars: interiors — turbulence

1. PROBLEMS IN MODELING STELLAR TURBULENT CONVECTION

Modeling heat transport by turbulence has been traditionally predicated on the assumption that it is possible to compute the heat flux with an expression reminiscent of the radiative flux. If w and θ are the turbulent velocity and temperature fields, one writes

$$\overline{w\theta} = -D_t \frac{\partial T}{\partial z}\,, \tag{1a}$$

where D_t is a turbulent heat diffusivity much larger than the kinematic one. For obvious reasons, equation (1a) is called a down gradient or Fickian approximation. Even at this formal level, without any specific form of D_t, there are theoretical and observational data that indicate that equation (1a) is seriously flawed. For example, even when $\partial T/\partial z > 0$, as it occurs in a stably stratified case, positive $\overline{w\theta}$ have been measured (Priestly & Swinbank 1947), thus exhibiting the phenomenon of *countergradient* rather than *downgradient* as in equation (1a). This can be seen in the following manner. Consider the first terms in the equations governing the fluctuating velocity and temperature fields (we do not employ the superadiabatic gradient for ease of notation; χ is the radiative conductivity),

$$\frac{\partial w}{\partial t} = g\alpha\theta - \frac{\partial p}{\partial z} + \cdots\,, \tag{1b}$$

$$\frac{\partial \theta}{\partial t} = -w\frac{\partial T}{\partial z} + \chi\frac{\partial^2\theta}{\partial z^2} + \cdots\,, \tag{1c}$$

where we have considered only the z-direction and where the dots represent the nonlinear terms to be discussed later. To construct the equation for $\overline{w\theta}$, one multiplies equation (1b) by θ, equation (1c) by w, and then averages and sums the two expressions. We have

$$\frac{\partial}{\partial t}\overline{w\theta} = -\overline{w^2}\frac{\partial T}{\partial z} + g\alpha\overline{\theta^2} - \overline{\theta\frac{\partial p}{\partial z}} + \cdots\,, \tag{1d}$$

which exhibits the presence of the positive potential energy term $\overline{\theta^2}$. Equation (1d) brings about two unknown variables, the temperature-pressure correlation, the last term in equation (1d), and the nonlinear term represented by the dots and which represents the diffusion of $\overline{w\theta}$, namely,

$$\cdots = -\frac{\partial}{\partial z}\overline{ww\theta}\ . \tag{1e}$$

By the same token, the equations for $\overline{w^2}$ and $\overline{\theta^2}$ that one constructs from equations (1b) and (1c) bring about pressure terms as well as diffusion terms. Since the fluctuating pressure p does not satisfy the hydrostatic equilibrium equation, the correlation terms

$$\overline{\theta\frac{\partial p}{\partial z}}\ ,\quad \overline{w\frac{\partial p}{\partial z}}\ , \tag{1f}$$

are among the most difficult variables for any turbulence model to describe, that is, to express in terms of the second-order moments. This is because, contrary to velocity third-order moments which exchange energy among eddies of different sizes, pressure forces tend to isotropize the components u^2, v^2, and w^2 of an eddy of a given size (Batchelor 1971). As a dynamical process, $\overline{w\,\partial p/\partial z}$ occurs on the dynamical timescale τ_{pv}, while $\overline{\theta\,\partial p/\partial z}$ occurs on a timescale $\tau_{p\theta}$. How are these timescales related to the dynamical time $\tau = K/\epsilon$ (K is the turbulent kinetic energy and ϵ is its rate of dissipation)? In most cases of geophysical turbulence, there are no internal processes that may become so dominant as to establish timescales shorter than τ, and it is thus generally assumed that

$$(\tau_{pv},\ \tau_{p\theta},\ \tau_\theta) \sim \tau\ . \tag{1g}$$

We have added τ_θ which is the dissipation timescale of the potential energy ($\sim \frac{1}{2}\overline{\theta^2}$), much as τ is the dissipation timescale of the kinetic energy. In stellar convection, one cannot use equation (1g) for the entire convective zone (CZ) since radiative processes become dominant near the borders with the radiative, stably stratified regions, where convection becomes inefficient and where a convection model is needed the most. In those regions, the radiative timescale dominates, and one may guess that instead of equation (1g) one should have

$$(\tau_{pv},\ \tau_{p\theta},\ \tau_\theta) \sim \mathrm{Pe}\ \tau < \tau\ , \tag{1h}$$

where Pe is the Peclet number, which is small when radiative effects are important. In the main part of the CZ, convection is very efficient, radiative losses are relatively unimportant, $\mathrm{Pe} > 1$ and equation (1g) is correct, but to treat the overshooting region one needs equation (1h). Actually, what one needs is a complete formula of the type

$$(\tau_{pv},\ \tau_{p\theta},\ \tau_\theta) = F(\mathrm{Pe})\tau\ , \tag{1i}$$

where one expects that for $\mathrm{Pe} > 1$, $F(\mathrm{Pe}) \sim \mathrm{Pe}^0$ and that for $\mathrm{Pe} < 1$, $F(\mathrm{Pe}) \sim \mathrm{Pe}$. We are not aware that any turbulence model has provided the function $F(\mathrm{Pe})$, and thus no existing model can be applied with confidence to the whole CZ, including the overshooting region. Using the above procedure, one of us (Canuto 1992) derived the dynamic equations for the five turbulence variables,

$$K\ ,\quad \tfrac{1}{2}\overline{w^2}\ ,\quad \tfrac{1}{2}\overline{\theta^2}\ ,\quad \overline{w\theta}\ ,\quad \epsilon\ , \tag{1j}$$

representing turbulent kinetic energy, turbulent kinetic energy in the z-direction, potential energy, convective flux, and rate of dissipation of turbulent kinetic energy. The model assumed equation (1g) which made it applicable only to the case of efficient convection, $\mathrm{Pe} > 1$. The present model is able to provide the function $F(\mathrm{Pe})$.

Let us now return to equation (1f). Following the physical picture discussed after equation (1f), one models the terms in equation (1f) as follows (Canuto 1992, 1993).

$$\overline{\theta\frac{\partial p}{\partial z}} = \tau_{p\theta}^{-1}\overline{w\theta} + c_1\, g\alpha\overline{\theta^2} + \cdots\ , \tag{2a}$$

$$\overline{w\frac{\partial p}{\partial z}} = \tau_{pv}^{-1}\left(\overline{w^2} - \frac{2}{3}K\right) + c_2\,\overline{\theta w} + \cdots\ , \tag{2b}$$

where the dots represent terms that may include mean shear, should the problem call for it. The rationale behind equation (2b) is that the restoring action of pressure forces trying to establish energy equipartition is directly proportional to the existing degree of anisotropy, the first term in equation (2b). The quantities $c_{1,2}$ are constants. Substitution of equation (2a) into equation (1d) gives

$$\frac{\partial}{\partial t}\overline{w\theta} = -\overline{w^2}\frac{\partial T}{\partial z} - \tau_{p\theta}^{-1}\overline{w\theta} + (1-c_1)g\alpha\overline{\theta^2} + \cdots\ , \tag{2c}$$

which though not yet complete, is sufficient for the moment. For example, in the stationary case, one has

$$\overline{w\theta} = -D_t\frac{\partial T}{\partial z} + (1-c_1)g\alpha\tau_{p\theta}\overline{\theta^2} + \cdots\ , \tag{2d}$$

$$D_t \equiv \tau_{p\theta}\overline{w^2}\ . \tag{2e}$$

How does equation (2d) compare with equation (1a)? First, equation (2d) exhibits the potential energy contribution which acts as a countergradient, and so it can explain the positive fluxes even when $\partial T/\partial z > 0$, whereas equation (1a) cannot. Second, the turbulence diffusivity D_t requires the knowledge of both $\overline{w^2}$, for which one must write a dynamic equation which entails the pressure correlation equation (2b), as well as of $\tau_{p\theta}$ which requires the knowledge of the function $F(\mathrm{Pe})$, equation (1i). Third, to evaluate $\overline{\theta^2}$ one must write the dynamic equation for $\overline{\theta^2}$ which in turn brings in the rate of dissipation ϵ_θ which entails the timescale τ_θ. In fact, from equation (1c) one derives.

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{\theta^2} = -\overline{w\theta}\frac{\partial T}{\partial z} + \chi\overline{\theta\frac{\partial^2\theta}{\partial z^2}}\ . \tag{2f}$$

Since the last term is rewritten as

$$\chi\overline{\theta\frac{\partial^2\theta}{\partial z^2}} = \frac{1}{2}\chi\frac{\partial^2}{\partial z^2}\overline{\theta^2} - \epsilon_\theta\ , \tag{2g}$$

$$\epsilon_\theta \equiv \chi\overline{\left(\frac{\partial\theta}{\partial z}\right)^2}\ , \tag{2h}$$

we are facing a new variable, ϵ_θ, the rate of dissipation of potential energy which must be modeled. We recall that the definition of ϵ is quite analogous (see eq. [5a]). In light of this, it does little good to write, as in the MLT, that

$$D_t = \tau_{p\theta}\overline{w^2} = \tau F(\mathrm{Pe})\overline{w^2} \sim w_{\mathrm{rms}}\, l \tag{2i}$$

since one simply hides all the difficulties in an ill-defined "mixing length" l. Rather one must consider the dynamic equations for $\overline{w^2}$, which one obtains from equations (1b) and (2b),

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{w^2} = (1-c_2)g\alpha\overline{\theta w} - \tau_{pv}^{-1}\left(\overline{w^2} - \frac{2}{3}K\right) + \cdots\ , \tag{3a}$$

This brings into the problem the timescale τ_{pv}. Equations (2c) and (3a) allow us to discuss the next important feature of convection, its nonlocal character, which is a phenomenon of primary importance in the treatment of the OV. When we insert back the nonlinear terms such as equation (1e) which represent a transport in physical space, we have, from equations (2c) and (3a),

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{w^2} + \frac{\partial}{\partial z}F_{\kappa e} = (1-c_2)g\alpha\overline{\theta w} - \tau_{pv}^{-1}\left(\overline{w^2} - \frac{2}{3}K\right), \quad (3b)$$

$$\frac{\partial}{\partial t}\overline{w\theta} + \frac{\partial}{\partial z}F_{w\theta} = -\overline{w^2}\frac{\partial T}{\partial z} - \tau_{p\theta}^{-1}\overline{w\theta} + (1-c_1)g\alpha\overline{\theta^2}, \quad (3c)$$

where $F_{\kappa e}$ and $F_{w\theta}$ represent the fluxes of kinetic energy and of temperature flux; that is,

$$F_{\kappa e} \equiv \tfrac{1}{2}\overline{w^3}, \quad F_{w\theta} \equiv \overline{w^2\theta}. \quad (3d)$$

When the eddies become deprived of the buoyant acceleration, they still overshoot into the stably stratified, radiative regions, where $\overline{w\theta}$ is negative. Since the right-hand side of equation (3b) is now negative, there is no local source of energy. A stationary solution can only exist if the terms involving equation (3d) act like a source: energy in brought in via a nonlocal, diffusion-like process. The dynamic equations for the variables (eq. [1j]) contain the terms,

$$\overline{Kw}, \quad \tfrac{1}{2}\overline{w^3}, \quad \overline{w\theta^2}, \quad \overline{w^2\theta}, \quad \overline{\epsilon w}, \quad (3e)$$

and one faces a rather severe challenge in constructing the required third-order moments (TOM), a problem that we discuss in § 8.

The OV extent depends crucially on the model one adopts for the TOMs. Recent nonlocal models (to be discussed in § 11) have adopted the following philosophy. If a second-order moment like $\overline{w\theta}$ can be related to a first-order moment like $\partial T/\partial z$ as in equation (1a), it is hoped that the TOM may be related to second-order moments via an extension of the down-gradient expression. Thus, one writes

$$\overline{Kw} \sim -D_t\frac{\partial K}{\partial z}, \quad \overline{w^3} \sim -D_t\frac{\partial}{\partial z}\frac{1}{2}\overline{w^2} \quad (4a)$$

$$\overline{\theta^2 w} \sim -D_t\frac{\partial}{\partial z}\overline{\theta^2}, \quad \overline{w^2\theta} \sim -D_t\frac{\partial}{\partial z}\overline{w\theta}. \quad (4b)$$

Since we have already shown that equation (1a) is not correct, any analogy based on it is bound to be doubtful. Indeed, recent large eddy simulations (Moeng & Wyngaard 1988) have shown that equation (4) underestimates the TOM by a factor of ~ 50. In § 11 we examine two recent nonlocal models which employ equation (4). Our suggestion is to abandon these phenomenological expressions and adopt the dynamic equations for the TOMs, which can be derived using a procedure analogous to the one employed to derive the second-order moments.

The final difficulty one encounters in constructing a turbulence model is the dissipation of kinetic energy ϵ. The key point is as follows. The nonlinear interactions distribute the available energy among eddies of different sizes so as to generate an equilibrium spectrum $E(k)$ versus k (the integral on all k' is K above). The process is called "transfer" specifically to underline the fact that it conserves energy: energy at the largest scales is the same as the energy that cascades to the smallest scales where it is dissipated into heat by kinematic molecular processes. *Energy conservation requires that ϵ cannot be zero.* The argument is manifestly independent of viscosity. Since the dissipation of the velocity field occurs via kinematic viscosity ν, the exact expression for ϵ can be derived directly from the basic Navier-Stokes equations (Landau & Lifshitz 1970),

$$\epsilon = 2\nu\overline{\left(\frac{\partial u_i}{\partial x_i}\right)^2} = 2\nu\Omega = 2\nu\int k^2E(k)dk, \quad (5a)$$

where Ω is the enstrophy $2\Omega = \overline{\omega^2}$ (ω is the vorticity) and $E(k)$ is the eddy energy spectrum. Since equation (5a) "seems" to depend on ν and since in stellar interiors ν is some 10 orders of magnitude smaller than the radiative conductivity, it is often implied that one can neglect ϵ since ν is small. This is, however, not so. The key point is that when $\nu \to 0$, $\Omega \to \infty$ and thus the product $\nu\Omega$ remains finite. The physical interpretation of equation (5a) is that ϵ is handed down by the large scales and whatever it is, the small scales will adjust their velocity spectra $E(k)$ and their sizes, the k^2 factor, to accommodate whatever is needed, the smaller the ν, the smaller the scales at which dissipation occurs. That ϵ is independent of ν has been proven many times in turbulence. The argument that $\nu \to 0$ is therefore mathematically and physically incorrect. Next comes the problem to describe ϵ. The most common practice is to assume that $E(k)$ can be described by the Kolomogorov spectrum

$$E(k) = \mathrm{Ko}\,\epsilon^{2/3}k^{-5/3}, \quad (5b)$$

where Ko is a constant 1.5–1.8. Integrating from $k_0 = \pi/l$, one obtains

$$\epsilon = c_\epsilon\frac{K^{3/2}}{l}, \quad c_\epsilon = \pi\left(\frac{2\mathrm{Ko}}{3}\right)^{3/2}. \quad (5c)$$

This expression has been employed in all nonlocal models we know of (Gough 1976; Xiong 1986; Balmforth 1992; Xiong, Cheng, & Deng 1997). Although equation (5c) exhibits no obvious pathologies, it has recently been found, quite unexpectedly, that it entails divergencies that may lead to lack of solutions of the whole system of equations (Canuto & Dubovikov 1997d). That alone counsels against the use of equation (5c). The divergences entailed by equation (5c) must have been averted thus far by tweaking the coefficients that appear in the equations. Even then, these nonlocal models usually produce a sizable overshoot, OV $\sim H_p$. Numerical simulations (Singh, Roxburgh, & Chan 1995) also predict large OV, but one must recall that computational limitations make it difficult to obtain reliable results. One is thus confronted with a situation in which both numerical simulations and theoretical models predict large OVs while observational data suggest small OV. Specifically, for massive stars OV $\leq 0.2H_p$ (Andersen, Nordstrom, & Clausen 1990; Stothers & Chin 1991; Shaller et al. 1992; Nordstrom, Andersen, & Andersen 1997; Kozhurina-Platais et al. 1997). In the solar case, helioseismological data yield OV $= 0.1H_p$ (Basu & Antia 1994) and OV $= 0.25H_p$ (Roxburgh & Vorontsov 1994). A recent analysis using data with lower error and improvements in the fitting procedure leads to OV $< 0.05H_p$ (Basu 1997). On the face of this *small versus large* OV dichotomy, it seems necessary to have a nonlocal theory of convection compatible with the data, a primary goal of this work.

Next, we would like to discuss the role of modeling versus numerical simulations. It is often erroneously believed that numerical simulations will render theories of convection unnecessary. We believe that just the opposite is true since, among other things, a theory is needed because of the results of numerical simulations. First, no computer today or in the foreseeable future is capable of simulating the $N \sim 10^{23}$ degrees of freedom that characterize a stellar convective zone ($N \sim \mathrm{Re}^{9/4}$, $\mathrm{Re} \sim 10^{10}$). One must model the unresolved scales, a task not without difficulties (Canuto 1997a). Assume, however, that we succeed. Still, the time requirements of the simulation codes makes it impossible to use them in a stellar code. On this basis alone, one concludes that a theory is needed. The second reason is more interesting and of deeper character. Consider the flux of turbulent kinetic energy and the convective flux (in units of $c_p \rho$)

$$F_{\kappa e} = \overline{Kw} \ , \quad F_c = \overline{w\theta} \ . \tag{6a}$$

Several LES studies (e.g., Stein & Nordlund 1989; Chan & Sofia 1989, 1996; Cattaneo et al. 1991) have demonstrated the following features: the convective region consists of well-organized, narrow (small filling factor), vigorous, downflows embedded in a midst of less organized, broad (large filling factors), more gentle upflows. If we denote the two components by subscripts d (down) and u (up), the results can be summarized as follows:

$$F^d_{\kappa e}, \text{ large}, \quad F^u_{\kappa e}, \text{ small} \ ; \tag{6b}$$

$$F^d_c, \text{ large}, \quad F^u_c, \text{ small} \ ; \tag{6c}$$

$$F^d_{\kappa e} + F^d_c \approx 0 \ , \tag{6d}$$

$$F^u_{\kappa e} + F^u_c \neq 0 \ . \tag{6e}$$

Equation (6d) tells us that the downward directed $F^d_{\kappa e}$ cancels almost exactly the convective flux F^d_c. In spite of the fact that both components in equation (6e) are small (the first is smaller than the other), *the less-organized motion is the only one that survives to transport the heat*. Thus, LES-DNS results have not made a theory of convection unnecessary; they have strengthened the need for it.

It is the primary goal of this paper to present a theory that avoids the shortcomings discussed above by using the most reliable turbulence modeling presently available. The novelties of the model are as follows:

1. Before being used for stellar convection, it was tested on a large variety of data from DNS, LES, laboratory, and geophysical flows (some of which exhibit quite vigorous convection), for a total of some 80 turbulence statistics (Canuto & Dubovikov 1996a, 1996b, 1996c, 1997a, 1997b, 1997c, [hereafter, Paper I, Paper II, Paper III, Paper IV, Paper V, Paper VI], 1997d).
2. The model provides the desired expression for all the timescales in equation (1i) versus Pe, equation (34).
3. It contains no mixing length since it employs a dynamic equation for ϵ, equations (35a) and (35b).
4. It can be formulated in k-space thus yielding the spectra of all the second-order moments, equations (16), as well as a one-point closure by integrating over all wavenumbers k's, equations (19). The standard Reynolds stress model is only given in the latter version and is unable to compute the timescale (eq. [1i]),
5. It is numerically manageable for it requires much less computer time than DNS-LES, and yet it matches their results.
6. It is possible to use it in a stellar code.

Having discussed the justification and the advantages of the model, we now discuss how the new model fits into the general scheme of turbulence modeling. In the last few decades, turbulence modeling has made considerable progress on two fronts.

1. *Two-point closure models.*—These are the most sought after models for they provide the maximum level of information, the spectra themselves, which are the foundations on which one constructs fluxes, kinetic energy, etc. The most widely known models are direct interaction approximation (DIA) and eddy damped quasi-normal markovian model (EDQNM) (Lesieur 1991). The latter has recently been used tio produce the logical successor of the MLT model (Canuto & Mazzitelli 1991 hereafter CM). One of the advantages of the CM model is that it has no adjustable parameters. A variety of tests attest to its validity (Stothers & Chin 1991, 1995; Basu & Antia 1994; Baturin & Miranova 1995; Antia & Basu 1997; Althaus & Benvenuto 1996; D'Antona & Mazzitelli 1996; Kupka 1996; Stothers & Chin 1997). It is, however, well known that DIA and EDQNM have met severe difficulties in dealing with inhomogeneities and thus with nonlocality (Leslie 1973), which is however, a primary feature of convection. This has prevented their use to describe real flows.

2. *One-point closure models.*—Here one foregoes the knowledge of the spectra being content with the integrals over all wavenumbers. The resulting model is known as Reynolds stress model (RSM). It has a long history of applications in engineering and geophysical turbulence, and its main features have been extended to stellar convection (Canuto 1992, 1993). There is, however, a feature that is unique to stellar interiors: radiative losses undercut the efficiency of convective transport and their influence must be fully accounted for, a situation that has no analog in geophysical flows. This consideration is all the more important when convection becomes less efficient and the temperature gradient ∇ is no longer adiabatic, as it occurs in the OV region where $\nabla - \nabla_{\mathrm{ad}} < 0$, while in the middle of the convective zone $\nabla - \nabla_{\mathrm{ad}} \sim 10^{-8}$. As of today, *no RSM can handle this feature*. We are able to solve the problem only because we adopt a new two-point closure model that, contrary to DIA and EDQNM, can successfully deal with inhomogeneities; we adopt the model and then integrate over all wavenumbers so as to produce a new RSM which no longer suffers from the limitations of the previous ones, and yet it remains manageable.

2. THE PHYSICAL CONTENT OF THE NEW MODEL

The starting point of the new model is the generally accepted view (Batchelor 1971; Monin & Yaglom 1975; McComb 1990; Lesieur 1991) that in fully developed turbulence there is a hierarchy of turbulent eddies which draw energy from the larger ones and cascade it without losses to the smaller ones. The latter group of eddies are therefore viewed as exerting an enhanced viscosity ν_t, called turbulent viscosity, which of course depends on the size of the eddy $\nu_t(k)$ with $\nu_t(\infty) = 0$ since the smallest eddies can only feel the kinematic viscosity ν. One thus introduces a dynamical viscosity $\nu_d(k)$ such that

$$\nu_d(k) \equiv \nu_t(k) + \nu = \int_k^\infty \Psi(p)dp + \nu \ . \tag{7a}$$

The integral expresses the fact that the enhanced viscosity felt by an eddy k is due to all the smaller eddies. The construction of $\Psi(p)$ is one of the major challenges of any turbulence theory, as documented by the large variety of phenomenological expressions (Monin & Yaglom 1975), beginning with the original one by Heisenberg. Because $\nu_t(k)$ represents the large k-part of the spectrum, it is also referred to as the UV (ultraviolet) component. We shall discuss our derivation of equation (7a) below. The construction of the other part, the action of the large, energy-containing eddies on an eddy k (by analogy, this part is referred to as the infrared part, IR) is considerably more difficult. It is in this context that our model introduces a major novelty since we suggest a model that is different from all the previous ones and the credibility of which (over and above the physical arguments on which it is based) is checked a posteriori by direct comparison with a host of data, as we shall discuss in § 4. Assuming for the moment that the UV and IR parts represent the main features of the interactions among eddies, one can think of splitting the nonlinear term in the original Navier-Stokes equations into two major components,

$$f_i^t(k)\ , \quad \nu_d(k)k^2u_i(k)\ , \tag{7b}$$

which represent the IR turbulent force and the UV parts, respectively. This physical picture of an eddy interacting with larger and smaller eddies, as represented by equation (7b) translates into the so-called Langevin type of equations. For an arbitrary eddy k, the turbulent velocity and the temperature fluctuating field are $u_i(\boldsymbol{k}, t)$ and $\theta(\boldsymbol{k}, t)$, we have

$$\frac{\partial}{\partial t}\, u_i(\boldsymbol{k}, t) = f_i^t(\boldsymbol{k}, t) - \nu_d(k, t)k^2u_i(\boldsymbol{k}, t) + f_i^{\text{ext}}(\boldsymbol{k}, t)\ , \tag{7c}$$

$$\frac{\partial}{\partial t}\, \theta(\boldsymbol{k}, t) = f_\theta^t(\boldsymbol{k}, t) - \chi_d(k, t)k^2\theta(\boldsymbol{k}, t) + f_\theta^{\text{ext}}(\boldsymbol{k}, t)\ , \tag{7d}$$

Here f_i^{ext} and f_θ^{ext} represent the true external forces that give rise to turbulence, while $\chi_d(k)$ is the thermal analog of $\nu_d(k)$; the kinematic ν and χ are included in ν_d and χ_d.

In Paper I it was shown that equation (7c) can be derived directly from the original Navier-Stokes equations. As usual in turbulence studies, derivations of this sort must use somewhat restrictive conditions to make the problem tractable. The hope is that the final results will exhibit a structure that is independent of the initial assumptions and can thus be viewed as of more general validity. Of course, this is not a proof, but the success of similar approaches in the context of critical phenomena (Wilson & Kogut 1974) is reassuring. As shown in detail in Paper I, one begins with the case of homogeneous and isotropic turbulence stirred by forces that, although not representative of real flows, have nonetheless a long history in turbulence studies since, among other things, they allow exact solutions of the NSE to be found (Foster, Nelson, & Stephen 1977; De Dominicis & Martin 1979). If, for convenience purposes, one represents the solution of the NSE in a diagrammatic form, as Wyld (1961) first did, one can sum up all the diagrams and present a formal solution for the energy spectrum which reveals that the nonlinear interactions have a twofold effect: they enhance the kinematic viscosity from ν to

$$\nu \to \nu_d(k) \equiv \nu + \nu_t(k)\ , \tag{8a}$$

and they affect the external forcing in such a manner that, if ϕ is the correlation function of such force, one has

$$\phi(k) \to \phi(k) + \tilde{\phi}(k)\ , \tag{8b}$$

indicating that the medium size eddies, although no longer forced directly by the external forcing ϕ, still feel a forcing $\tilde{\phi}$ which is ultimately responsible for the existence of the non-equilibrium Kolmogorov spectrum. This is often called the Wyld-Dyson result in analogy with the fact that in electrodynamics Dyson derived the exact equations for the electron and the photon. In them, the electron bare mass is renormalized by a mass operator (an analog of $\tilde{\phi}$), while the vacuum acquires a dielectric constant larger than unity, the polarization tensor, an analog of $\nu_d(k)$. The DIA model mentioned earlier corresponds to a particular subset of Wyld diagrams, while the EDQNM model which is frequently employed in turbulence studies is a simpler form of DIA and thus carries with it the same limitations (diagram wise) of the DIA.

Stochastic, Langevin-type, equations have a long history in turbulence modeling. Kraichnan (1970) considered this equation both in general terms and as it relates to DIA. Leith (1971) independently suggested a very similar model. Herring & Kraichnan (1971) presented a detailed review of Langevin-type equations corresponding to different closures, as well as new calculations to compare the peformance of these models vis-a-vis one another, numerical simulations and laboratory data. The same type of equation was also suggested as a tool to construct a SGS (subgrid scale) model to be employed in large eddy simulations. The suggestion was pursued by Chasnov (1991).

However, equations (7c) and (7d) lack predictive power until they are supplemented with a specific model to compute $\nu_d(k)$, $\chi_d(k)$ and the work of the turbulent forces f^t values.

3. CONSTRUCTION OF THE MODEL

3.1. *Velocity Field*

We adopt the generally accepted view that in fully developed turbulence, the transfer in k-space of turbulent energy is a local phenomenon that takes place primarily among eddies of similar sizes or ks. In analogy with the case of a gas flow characterized by a current $j(x)$;

$$j(x) = \rho(x)v(x)\ , \tag{9a}$$

we propose to characterize the "propagation" of energy in k-space by a current $\Pi(k)$, an energy density $E(k)$ and a rapidity $r(k)$. With the mapping

$$j \to \Pi(k)\ , \quad \rho \to E(k)\ , \quad v \to r(k)\ , \tag{9b}$$

we have

$$\Pi(k) = r(k)E(k)\ . \tag{9c}$$

In principle, of course $r(k)$ is a functional of $E(k)$ and thus equation (9c) is quite nonlinear. The next problem is to relate $r(k)$ to $E(k)$. To that end, let us multiply equation (7c) by $u_i(\boldsymbol{k}', t)$. Since by definition

$$E(k, t) = \tfrac{1}{2}k^2 \int d\Omega_{\boldsymbol{k}}\, dk' \langle u_i(\boldsymbol{k}, t)u_i(\boldsymbol{k}', t)\rangle\ , \tag{9d}$$

the dynamic equation for $E(k)$ is

$$\frac{\partial}{\partial t}\, E(k) = A_t(k) - 2k^2\nu_d(k)E(k) + A_{\text{ext}}\ , \tag{9e}$$

where $A_t(k)$ is the work performed by the turbulent force $\boldsymbol{f}^t$

$$A_t(k) = k^2 \int d\Omega_{\boldsymbol{k}}\, dk' \langle u_i(\boldsymbol{k}', t) f_i^t(\boldsymbol{k}, t) \rangle \,. \tag{9f}$$

$A_{\rm ext}$ has an analogous expression with $\boldsymbol{f}^{\rm ext}$ in lieu of $\boldsymbol{f}^t$. On the other hand, it is known (Batchelor 1971) that the dynamic equation for $E(k)$ can be written quite generally as

$$\frac{\partial}{\partial t} E(k) + 2\nu k^2 E(k) = -\frac{\partial}{\partial k} \Pi(k) + A_{\rm ext} \,. \tag{9g}$$

The terms corresponding to the molecular viscosity and external forces do not require an explanation. The first term on the right-hand side represents the nonlinear interactions in the Navier-Stokes equations that conserve energy. They appear as a divergence, and so the integral in physical space over the whole volume vanishes. In k-space, this is expressed as the k-divergence of a flux $\Pi(k)$ so that the integral over all k's vanishes. The term is also known as the transfer $T(k)$;

$$T(k) = -\frac{\partial}{\partial k} \Pi(k) \,, \quad \int T(k) dk = 0 \,. \tag{9h}$$

Needless to say, the major challenge of any turbulence mode is the transfer $T(k)$. Since we have introduced equation (9c), we also have

$$T(k) = -r(k) \frac{\partial}{\partial k} E(k) - \left[\frac{\partial}{\partial k} r(k)\right] E(k) \,, \tag{9i}$$

which allows us to write

$$T_{\rm UV} = -E(k) \frac{\partial}{\partial k} r(k) \,, \quad T_{\rm IR} = -r(k) \frac{\partial}{\partial k} E(k) \,. \tag{9j}$$

Comparison with equation (9e) yields

$$A_t(k) = -r(k) \frac{\partial}{\partial k} E(k) \,, \quad \nu_d(k) = \nu + \frac{1}{2} k^{-2} \frac{\partial}{\partial k} r(k) \,, \tag{9k}$$

which express A_t and $\nu_d(k)$ in terms of $r(k)$, which remains the only undetermined variable. Once it it expressed in terms of $E(k)$, equation (9e) is complete and the spectrum $E(k)$ can be derived for any given external forcing. Inverting the second formula given in equation (9k), we have

$$r(k) = 2 \int_0^k p^2 \nu_t(p) dp \,. \tag{9l}$$

The problem thus comes down to finding an expression for $\nu_d(k)$. This is a classical problem in turbulence theory, and several models have been proposed to compute it. Canuto, Goldman, & Chasnov (1988) derived the relation

$$\nu_d(k) = \left[\nu^2 + \gamma \int_k^\infty p^{-2} E(p)\right]^{1/2} , \tag{10a}$$

but the constant γ could not be determined from within the model. As shown in Paper I, use of RNG (renormalization group) techniques leads to the expression

$$\nu_d(k) = \left[\nu^2 + \tfrac{2}{5} \int_k^\infty p^{-2} E(p)\right]^{1/2} , \tag{10b}$$

which has the same structure as equation (10a) but has the advantage that the unknown γ is now uniquely determined. Also discussed in Paper I is the fact that equation (10b) generalizes previous RNG expressions for ν_d which were only valid for restricted cases. The dynamic equation for the spectrum $E(k)$ is now complete: A_t is given by equation (9k), $r(k)$ is given by equation (9l), and $\nu_d(k)$ is given by equation (10b). Finally, we recall that the turbulent kinetic energy K is obtained from

$$K = \int E(k) dk \,. \tag{10c}$$

In summary, we have succeeded in relating both the UV and the IR components of the nonlinear transfer to a unique function, the rapidity $r(k)$, which in turn depends on the turbulent viscosity $\nu_t(k)$.

3.2. *Temperature Field*

An analogous procedure leads to the equation for the spectrum $E_\theta(k)$ of the temperature variance

$$E_\theta = \tfrac{1}{2}\overline{\theta^2} = \int E_\theta(k) dk \,. \tag{11a}$$

We have (see Paper I for details)

$$\frac{\partial}{\partial t} E_\theta(k) = A_t^\theta(k) - 2k^2 \chi_d(k) E(k) + A_{\rm ext}^\theta \,, \tag{11b}$$

where

$$A_t^\theta(k) = -r_\theta(k) \frac{\partial}{\partial k} E_\theta(k) \,, \tag{11c}$$

$$r_\theta(k) = E_\theta^{-1}(k) \Pi_\theta(k) = 2 \int_0^k p^2 \chi_t(p) dp \,. \tag{11d}$$

3.3. *Momentum and Temperature Diffusivities $\nu_d(k)$ and $\chi_d(k)$*

The RNG technique that led to equation (10b) also shows the derivation of the differential equations that yields $\chi_d(k)$ in terms of $\nu_d(k)$,

$$\frac{d}{d\nu_d} \chi_d = \beta^{-1} \nu_d (\nu_d + \chi_d)^{-1} \,, \tag{11e}$$

with the boundary condition $\chi_d(\nu) = \chi$, and constant $\beta = 0.3$. The analytic solution of equation (11e) is

$$b\chi_t = \nu_t + (b\chi - \nu)\left[\left(1 + \frac{a\chi_t + \nu_t}{a\chi + \nu}\right)^{-a/b} - 1\right], \tag{11f}$$

where χ_t and ν_t depend on k and where $2a = (\beta^2 + 4\beta)^{1/2} - \beta$, $b = a + \beta$.

4. TESTS OF THE MODEL

4.1. *Inertial Range Spectra*

When the fluxes Π and Π_θ are constant, the model ought to reproduce well-known inertial spectra. Indeed, when

$$\frac{\partial}{\partial t} = 0 \,, \quad A_{\rm ext} = 0 \,, \quad \Pi(k) = \epsilon \,, \tag{12a}$$

Equation (9e) yields the well-known Kolomogorov spectrum,

$$E(k) = \tfrac{5}{3} \epsilon^{2/3} k^{-5/3} \,. \tag{12b}$$

The model also predicts the Kolmogorov constant Ko $= 5/3$, in very good agreement with recent data $1.59 <$ Ko < 1.88 (Praskovsky & Oncley 1994). As for the temperature field, when

$$\Pi_\theta = \epsilon_\theta \,, \tag{12c}$$

the model yields the well-known Corrsin-Obukhov spectrum for the inertial-convective regime (Monin & Yaglom

1975; Lesieur 1991)

$$E_\theta(k) = \mathrm{Ba}\, \epsilon_\theta \epsilon^{-1/3} k^{-5/3}\ , \quad \mathrm{Ba} = \mathrm{Ko}\, \sigma_t\ , \tag{12d}$$

The Batchelor constant Ba is predicted to be the product of Ko and the turbulent Prandtl number $\sigma_t = 0.72$.

4.2. *Homogeneous Flows*

In Paper III, the model predictions were tested against DNS, LES, and laboratory data on dissipation region, free decay for both energy and temperature fields, skewness, and the inertial-conductive regime. The model results compare very well with the data.

4.3. *Extension to Anisotropic Flows*

The above tests refer to homogeneous and isotropic flows. Next the model was extended to deal with anisotropic but still homogeneous flows. A large set of DNS and LES data are available for shear driven flows for plain strain, axisymmetric contraction, and homogeneous strain. The data provide the total turbulent kinetic energy. Reynolds stresses, dissipation rate tensor, and the pressure-velocity correlation tensor discussed in § 1. The performance of the model is in all cases very satisfactory (Paper III).

4.4. *Extension to Inhomogeneous Flows: Convection*

The model was applied to the case of turbulent convection for which a large set of laboratory and DNS data has recently become available (Paper IV). Specifically, Nusselt number Nu versus Ra (Rayleigh number), $\overline{\theta_w^2}$, $\overline{\theta_c^2}$ versus Ra (temperature variance near the wall and at the center of the convective cell), λ_T versus Ra (thermal boundary layer thickness), z-profile of mean temperature T, horizontal and vertical Peclet numbers versus Ra, spectra of total kinetic energy, vertical kinetic energy, temperature variance and temperature flux versus k_h (horizontal wavenumber), dependence of the Nu versus Ra relation on the molecular Prandtl number. In all cases, the model performance is good.

4.5. *Two-dimensional and Rotating Turbulence*

In Paper VI, the model was shown to predict well known features of two-dimensional turbulence (enstrophy and energy inertial regimes) and of rotating turbulence. It explains for the first time the LES data of the very different growth exhibited by vertical and horizontal length scales, a feature that has remained unexplained until this model become available.

In conclusion, we are not aware of any other turbulence model that was submitted to such a large number of tests and which had performed equally well, if one further considers that the model has no free parameters.

5. STELLAR CONVECTION: SPECTRAL DYNAMIC EQUATIONS

In the Boussinesq approximation, the external forces appearing in the Langevin equations (7c) and (7d) are given by

$$f_i^{\mathrm{ext}}(\boldsymbol{k}) = -\alpha P_{ij}(\boldsymbol{k}) g_j \theta(\boldsymbol{k})\ , \tag{13a}$$

$$f_\theta^{\mathrm{ext}}(\boldsymbol{k}) = \beta_i u_i(\boldsymbol{k})\ , \quad \beta_i = -\frac{\partial T}{\partial x_i} - \frac{g_i}{c_p}\ , \tag{13b}$$

where $P_{ij} = \delta_{ij} - k^{-2} k_i k_j$ is the projection operator, α is the thermal expansion coefficient, and $\boldsymbol{g}$ is the gravitational acceleration. The final form of the dynamic equations reads:

$$\frac{\partial}{\partial t} u_i(\boldsymbol{k}, t) = -\alpha P_{ij}(\boldsymbol{k}) g_j \theta(\boldsymbol{k}) + f_i^t(\boldsymbol{k}, t) - k^2 \nu_d(k, t) u_i(\boldsymbol{k}, t)\ , \tag{13c}$$

$$\frac{\partial}{\partial t} \theta(\boldsymbol{k}, t) = \beta_i u_i(\boldsymbol{k}) + f_\theta^t(\boldsymbol{k}, t) - k^2 \chi_d(k, t) \theta(\boldsymbol{k}, t)\ . \tag{13d}$$

To proceed, we begin by defining the densities of turbulent kinetic energy, temperature variance, temperature flux, and z-component of the turbulent kinetic energy:

$$\delta(\boldsymbol{k} + \boldsymbol{k}') e(\boldsymbol{k}, t) = \tfrac{1}{2} \langle u_i(\boldsymbol{k}', t) u_i(\boldsymbol{k}, t) \rangle\ , \tag{14a}$$

$$\delta(\boldsymbol{k} + \boldsymbol{k}') e_\theta(\boldsymbol{k}, t) = \tfrac{1}{2} \langle \theta(\boldsymbol{k}', t) \theta(\boldsymbol{k}, t) \rangle\ , \tag{14b}$$

$$\delta(\boldsymbol{k} + \boldsymbol{k}') j(\boldsymbol{k}, t) = \langle w(\boldsymbol{k}', t) \theta(\boldsymbol{k}, t) \rangle\ , \tag{14c}$$

$$\delta(\boldsymbol{k} + \boldsymbol{k}') e_z(\boldsymbol{k}, t) = \tfrac{1}{2} \langle w(\boldsymbol{k}', t) w(\boldsymbol{k}, t) \rangle\ , \tag{14d}$$

From these densities, one then constructs the corresponding spectra,

$$E(k) = k^2 \int d\Omega_{\boldsymbol{k}}\, e(\boldsymbol{k})\ , \quad E_\theta(k) = k^2 \int d\Omega_{\boldsymbol{k}}\, e_\theta(\boldsymbol{k})\ , \tag{15a}$$

$$J(k) = k^2 \int d\Omega_{\boldsymbol{k}}\, j(\boldsymbol{k})\ , \quad E_z(k) = k^2 \int d\Omega_{\boldsymbol{k}}\, e_z(\boldsymbol{k})\ , \tag{15b}$$

with

$$\overline{w\theta} \equiv J = \int_0^\infty J(k) dk\ , \tag{15c}$$

$$\tfrac{1}{2}\overline{w^2} = \int_0^\infty E_z(k) dk\ . \tag{15d}$$

The kinetic energy K and the temperature variance have already been defined in equations (10c) and (11a). Multiplying equation (13c) by $u_i(\boldsymbol{k}')$, w and θ, respectively, and averaging, we obtain the equations for the densities in equation (14):

$$\frac{\partial}{\partial t} e + D_f(e) = a^t + \alpha g j - 2k^2 \nu_d e + \frac{\partial}{\partial z}\left(\nu \frac{\partial e}{\partial z}\right), \tag{16a}$$

$$\frac{\partial}{\partial t} e_\theta + D_f(e_\theta) = a_\theta^t + \beta j - 2k^2 \chi_d e_\theta + \frac{\partial}{\partial z}\left(\chi \frac{\partial e_\theta}{\partial z}\right), \tag{16b}$$

$$\frac{\partial}{\partial t} j + D_f(j) = 2\alpha g P_{zz} e_\theta + 2\beta e_z - k^2(\nu_d + \chi_d) j + \frac{1}{2} \frac{\partial}{\partial z} (\nu + \chi) \frac{\partial j}{\partial z}, \tag{16c}$$

$$\frac{\partial}{\partial t} e_z + D_f(e_z) = \frac{1}{2} P_{zz} a^t + \alpha g P_{zz} j - 2k^2 \nu_d e_z + \frac{\partial}{\partial z}\left(\chi \frac{\partial e_z}{\partial z}\right), \tag{16d}$$

where a^t and a_θ^t represent the work by the turbulent forces

$$\delta(\boldsymbol{k} + \boldsymbol{k}') a^t(\boldsymbol{k}, t) = \langle f_i^t(\boldsymbol{k}', t) u_i(\boldsymbol{k}, t) \rangle\ , \tag{17a}$$

$$\delta(\boldsymbol{k} + \boldsymbol{k}') a_\theta^t(\boldsymbol{k}, t) = \langle f_\theta^t(\boldsymbol{k}', t) \theta(\boldsymbol{k}, t) \rangle\ , \tag{17b}$$

As discussed in § 3, they are given in terms of the energy and temperature fluxes $\Pi(k)$ and $\Pi_\theta(k)$ via the relations

$$a^t = (4\pi k^2)^{-1} A^t(k) = -(4\pi k^2)^{-1} \frac{E'(k)}{E(k)} \Pi(k) , \quad (18a)$$

$$a^t_\theta = (4\pi k^2)^{-1} A^t_\theta(k) = -(4\pi k^2)^{-1} \frac{E'_\theta(k)}{E_\theta(k)} \Pi_\theta(k) , \quad (18b)$$

Furthermore, the fluxes' Π values are related to the dynamical viscosities/conductivities by equations (9c), (9l), and (11d); ν_t and χ_t are related via equations (11e) and (11f) while $\nu_d(k)$ is given by equation (10b). The diffusion terms D_f to be specified later represent the fact that the flow is inhomogeneous, while the last term in each equation represents the losses due to kinematic processes. The turbulence equations (16a)–(16d) must be supplemented by the dynamic equation governing the evolution of the mean temperature field

$$c_p \rho \frac{\partial T}{\partial t} + \frac{\partial K}{\partial t} = -\frac{\partial}{\partial z} (F_r + F_c + F_{\kappa e}) , \quad (18c)$$

where F_r and $F_{\kappa e}$ represent the radiation flux and the flux of turbulent kinetic energy, to be specified in what follows.

6. STELLAR CONVECTION: ONE-POINT DYNAMIC EQUATIONS

Equations (16a)–(18c) have recently been solved for the case of laboratory convection at high Ra (Paper IV). By solving them, one obtains the spectra $E(k)$, $E_z(k)$, $E_\theta(k)$, and $J(k)$ with which one computes all the required statistics. The method employed to solve numerically these equations is presented in detail in Paper IV. The same procedure can in principle be employed in the case of stellar convection. However, since the Prandtl number is much smaller (in the sun $\mathrm{Pr} \sim 10^{-10}$) than in laboratory ($\mathrm{Pr} \sim 1$) and the number of wavenumbers (scales) to be included is considerably larger, we propose to first integrate equations (16a)–(16d) over all k and then use them in the stellar case. By so doing, we shall also be able to compare the ensuing equations with those of the Reynolds stress method (RSM).

6.1. *Reynolds Stress Model*

The dynamic equations for the variables (eq. [1j]) are given by (Canuto 1992, 1993),

$$\frac{\partial K}{\partial t} + D_f = g\alpha J - \epsilon , \quad (19a)$$

$$\frac{\partial}{\partial t} \frac{1}{2} \overline{\theta^2} + D_f = \beta J - \epsilon_\theta + \frac{1}{2} \frac{\partial}{\partial z}\left(\chi \frac{\partial}{\partial z} \overline{\theta^2}\right) , \quad (19b)$$

$$\frac{\partial}{\partial t} J + D_f = \beta \overline{w^2} + (1-\gamma_1) g\alpha \overline{\theta^2} - \tau_{p\theta}^{-1} J + \frac{1}{2} \frac{\partial}{\partial z}\left(\chi \frac{\partial}{\partial z} J\right) , \quad (19c)$$

$$\frac{\partial}{\partial t} \frac{1}{2} \overline{w^2} + D_f = -\tau_{pv}^{-1}\left(\overline{w^2} - \frac{2}{3} K\right) + \frac{1}{3} (1 + 2\beta_5) g\alpha J - \frac{1}{3} \epsilon . \quad (19d)$$

With equations (16a)–(16d), the knowledge of the spectral functions allows one to compute all the statistics, in particular ϵ and ϵ_θ. In the RSM model, the lack of such spectra forces one to model ϵ and ϵ_θ which we write as

$$\epsilon = 2K\tau^{-1} , \quad \epsilon_\theta = \overline{\theta^2}\tau_\theta^{-1} . \quad (20a)$$

The RSM also contains two parameters, γ, β and the timescales,

$$\tau_\theta , \quad \tau_{p\theta} , \quad \tau_{pv} . \quad (20b)$$

The standard RSM is unable to compute either the constants or equation (20b). This is no longer the case with the new model, as we now show.

6.2. *New Reynolds Stress Model*

To obtain the RSM equations from our model, we first integrate equation (16a) over $d\Omega_{\boldsymbol{k}}$ and make use of (15a) and (18a). Since $\nu_t \gg \nu$, we obtain

$$\frac{\partial}{\partial t} E - \frac{\partial}{\partial z}\left(\nu_t \frac{\partial E}{\partial z}\right) = A^t(k) - 2k^2 \nu_t E + \alpha g J - 2k^2 \nu E , \quad (21a)$$

where $E \equiv E(k)$, $J \equiv J(k)$, $\nu_t \equiv \nu_t(k)$. Next, we recall that because of equations (9h) and (9i) and the first of (9k), we have

$$\int T(k) dk = \int dk[A^t(k) - 2k^2 \nu_t(k) E(k)] = 0 . \quad (21b)$$

Integrating over k, equation (21a) becomes identical to equation (19a) with

$$\epsilon = 2\nu \int k^2 E(k) dk . \quad (21c)$$

Using the same procedure, equation (16b) becomes identical to (19b), with

$$\epsilon_\theta = 2\chi \int k^2 E_\theta(k) dk . \quad (21d)$$

Next, we consider equation (16c) and carry out the same procedure. We recover equation (19c) with the identification

$$1 - \gamma_1 = \bar{P}^\theta_{zz} \equiv \int P_{zz}(k, k_z) e_\theta(k) dk \left[\int e_\theta(k) dk\right]^{-1} , \quad (21e)$$

$$\tau_{p\theta}^{-1} = \int k^2 (\nu_d + \chi_d) j(k) dk \left[\int J(k) dk\right]^{-1} . \quad (21f)$$

Next, we integrate equation (16d). Using the first formula of equation (18a), we obtain

$$\int a^t(k) P_{zz}\, dk = \tfrac{2}{3} \int A_t(k) dk . \quad (22a)$$

Thus, with $\nu_t \gg \nu$, we obtain

$$\frac{\partial}{\partial t} \frac{1}{2} \overline{w^2} + D_f = \frac{1}{3} \int A_t(k) dk + \alpha g \int d\boldsymbol{k} P_{zz} j - 2 \int k^2 \nu_d E_z(k) dk . \quad (22b)$$

Furthermore, using equation (21b) we rewrite equation (22b) as

$$\frac{\partial}{\partial t} \frac{1}{2} \overline{w^2} + D_f = -2 \int k^2 \nu_t(k)\left(E_z - \frac{1}{3} E\right) dk + \alpha g \int d\boldsymbol{k} P_{zz} j - 2\nu \int k^2 E_z(k) dk . \quad (22c)$$

The first integral differs from zero only for the smallest k since at large k, eddies become isotropic and $E_z \to 1/3E$. We can thus take $k^2\nu_t(k)$ out of the integral and evaluate it at $k = k_0$. By the same token, since the last integral peaks at large k we can take $3E_z \to E$. Thus we obtain from equation (22c),

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{w^2} + D_f = -k_0^2\nu_t(k_0)\left(\overline{w^2} - \frac{2}{3}K\right) + \alpha g J\langle P_{zz}\rangle_j - \frac{1}{3}\epsilon\,, \tag{23a}$$

where

$$\bar{P}_{zz}^j \equiv J^{-1}\int dk P_{zz}\, j = \left[\int dk j(k)\right]^{-1}\int dk P_{zz}\, j(k)\,. \tag{23b}$$

Comparing this now with equation (19d), we derive the relations

$$\tau_{pv}^{-1} = k_0^2\nu_t(k_0)\,, \quad \tfrac{1}{3}(1 + 2\beta_5) = \bar{P}_{zz}^j\,. \tag{23c}$$

Although equations (21e) and (23b) can be computed only if one knows the spectral functions, reliable results can be obtained by using the approximation

$$P_{zz}^j \sim \bar{P}_{zz}^\theta = \tfrac{2}{3}\,, \tag{23d}$$

which implies that

$$\gamma_1 = \tfrac{1}{3}\,, \quad \beta_5 = \tfrac{1}{2}\,. \tag{23e}$$

These predictions compare well with the empirical values (Appendix A in Canuto 1993)

$$\gamma_1 = \tfrac{1}{4}\,, \quad \beta_5 = \tfrac{7}{10} \tag{23f}$$

6.3. *Timescales* τ_{pv}, $\tau_{p\theta}$, *and* τ_θ

We assume a Kolomogorov inertial energy spectrum with a cutoff k_0 (H is the Heaviside function)

$$E(k) = \text{Ko}\,\epsilon^{2/3}k^{-5/3}H(k - k_0)\,. \tag{24a}$$

From the general results of equations (10b) and (10c) we derive,

$$\nu_t \equiv \nu_t(k_0) = \left(\frac{3\,\text{Ko}}{20}\right)^{1/2}\epsilon^{1/3}k_0^{-4/3}\,, \quad K = \frac{3}{2}\text{Ko}\,\epsilon^{2/3}k_0^{-2/3}\,. \tag{24b}$$

Combining them, we obtain

$$\nu_t = C_\mu \frac{\text{K}}{\epsilon}\,, \tag{24c}$$

where the constant C_μ is given by

$$C_\mu = (10\pi^2)^{-1/2}c_\epsilon\,, \quad c_\epsilon \equiv \pi\left(\frac{2}{3\text{Ko}}\right)^{3/2}\,. \tag{24d}$$

Using the first of equations (20a) and (23c), we obtain

$$\frac{\tau}{\tau_{pv}} = \left(\frac{27}{20}\text{Ko}^3\right)^{1/2} = \frac{5}{2}\,, \tag{25a}$$

for Ko = 5/3. Equation (25a) compares favorably with the empirical estimate (Canuto 1993).

Next, we compute $\tau_{p\theta}$ which, contrary to τ_{pv}, depends on the efficiency of convection characterized by $\chi_t > \chi$ (efficient turbulence) and $\chi_t < \chi$ (inefficient turbulence). The Peclet number Pe is often used to characterize the two regimes. Because of the absence of a precise definition (a numerical constant can always be inserted without changing the meaning of Pe), we prefer to distinguish the two regimes with the ratio χ_t/χ. A reasonable (but not unique) definition of Pe is (τ_χ is the radiative timescale)

$$\text{Pe} = \frac{\tau_\chi}{\tau} = \frac{1}{2}C_\mu^{-1}c_\epsilon^2\frac{\nu_t}{\chi}\,,$$

$$\tau_\chi = \frac{l^2}{\chi}\,, \quad \tau = \frac{2}{\epsilon}K\,. \tag{26a}$$

When $\chi_t > \chi$, $E(k)$, $E_\theta(k) \sim k^{-5/3}$, and $e(k)$, $e_\theta(k) \sim k^{-11/3}$. From equation (16c) we have that near stationarity,

$$k^2(\nu_d + \chi_d)j(k) \sim k^{-11/3}\,. \tag{26b}$$

Since we are dealing with $\nu_t \gg \nu$, equation (11e) simplifies to

$$\frac{d}{d\nu_t}\chi_t = \gamma^{-1}(1 + \sigma_t^{-1})^{-1}\,, \tag{26c}$$

or, integrating,

$$\chi_t = \gamma^{-1}(1 + \sigma_t^{-1})^{-1}\nu_t\,,$$

$$\sigma_t \equiv \frac{\nu_t}{\chi_t} = \frac{1}{2}[\gamma + (\gamma^2 + 4\gamma)^{1/2}]\,. \tag{26d}$$

Since $\nu_t(k) \sim k^{-4/3}$ and since

$$\nu_d + \chi_d \to \nu_t + \chi_t = \nu_t(1 + \sigma_t^{-1})\,, \tag{26e}$$

we have from equation (26b) $j(k) \sim k^{-13/3}$, $J(k) \sim k^{-7/3}$ and equation (21f) gives

$$\tau_{p\theta}^{-1} = (\tfrac{3}{5}\text{Ko})^{1/2}(1 + \sigma_t^{-1})k_0^{2/3}\epsilon^{1/3}\,. \tag{27a}$$

Since $k_0^{2/3}\epsilon^{1/3} = 3\,\text{Ko}\,\tau^{-1}$, we finally have

$$\frac{\tau}{\tau_{p\theta}} = \left(\frac{27}{5}\text{Ko}^3\right)^{1/2}(1 + \sigma_t^{-1})\,. \tag{27b}$$

For Ko = 5/3, the coefficient becomes $5(1 + \sigma_t^{-1})$ to be compared with the empirical value $f_1 = 7.5$ (Canuto 1993, eq. [81] and Appendix A). Using equations (12d) and (11a), we obtain

$$\tfrac{1}{2}\overline{\theta^2} = \tfrac{3}{2}\text{Ba}\,\epsilon_\theta\epsilon^{-1/3}k_0^{-2/3} = \tfrac{1}{2}\sigma_t\epsilon_\theta\tau\,, \tag{28a}$$

so that

$$\frac{\tau_\theta}{\tau} = \sigma_t\,. \tag{28b}$$

When, on the other hand, $\chi > \chi_t$, equation (11e) becomes to first order in Pe,

$$\frac{d}{d\nu_t}\chi_t = \gamma^{-1}\frac{\nu_t}{\chi}\,, \quad \chi_t = \frac{1}{2\gamma}\frac{\nu_t^2}{\chi}\,. \tag{29a}$$

Since in this case

$$\nu_d + \chi_d \to \nu_t + \chi \to \chi\,, \tag{29b}$$

we derive $j(k) \sim k^{-17/3}$, $J(k) \sim k^{-11/3}$ and so

$$\tau_{p\theta} = (4\chi k_0^2)^{-1} = (4\pi^2)^{-1}l^2\chi^{-1}\,, \tag{30a}$$

$$\frac{\tau}{\tau_{p\theta}} = \frac{4\pi^2}{\text{Pe}}\,. \tag{30b}$$

In going from efficient to inefficient convection, $\tau_{p\theta}$ changes from $\sim\tau$ to $\sim\tau\text{Pe}$. Since the latter is smaller than τ, it implies a much stronger damping of the convective flux by the pressure-temperature correlations, as from the third term in equation (19c).

Finally, we compute τ_θ. Consider equation (16b) in a stationary regime. Since

$$\chi > \nu_t > \chi_t \sim \chi^{-1}\nu_t^2 , \tag{31a}$$

we have

$$a_\theta^t \sim k^2\chi_t e_\theta \ll k^2\chi e_\theta , \tag{31b}$$

which, after integration over $k^2 d\Omega_{\boldsymbol{k}}$, leads to

$$\beta J(k) = 2k^2\chi E_\theta(k) . \tag{31c}$$

As we show below, the first term in the right-hand side in equation (16c) is smaller than the second term, so that after integration over space angles we obtain

$$2\beta E_z(k) = k^2\chi J(k) . \tag{32a}$$

Thus,

$$E_\theta(k) = \beta^2\chi^{-2}k^{-4}E_z(k) = \tfrac{1}{3}\,\mathrm{Ko}\;\beta^2\chi^{-2}\epsilon^{2/3}k^{-17/3} , \tag{32b}$$

where we have used $E_z(k) = \frac{1}{3}E(k)$ and the Kolmogorov spectrum. It follows that

$$\frac{1}{2}\,\overline{\theta^2} = \frac{\mathrm{Ko}}{14}\,\beta^2\chi^{-2}\epsilon^{2/3}k_0^{-14/3} , \quad \epsilon_\theta = \frac{7}{4}\,\chi k_0^2\,\overline{\theta^2} . \tag{32c}$$

From equation (20a), we finally derive that

$$\frac{\tau_\theta}{\tau} = \left(\frac{4}{7\pi^2}\right)\mathrm{Pe} < 1 , \tag{33}$$

a situation analogous to that of $\tau_{p\theta}$. In summary, the expressions for $\chi > \chi_t$ and $\chi_t > \chi$ are

$$\tau_{pv} = \tfrac{2}{5}\tau , \tag{34a}$$

$$\frac{\tau_{p\theta}}{\tau} = \frac{1}{4\pi^2}\,\mathrm{Pe}\left(1 + \frac{1}{2}\nu_t\chi^{-1} + \frac{1}{2}\chi_t\chi^{-1}\right)^{-1} , \tag{34b}$$

$$\frac{\tau_\theta}{\tau} = \frac{4}{7\pi^2}\,\mathrm{Pe}\left(1 + \frac{8}{35}\,\chi_t\chi^{-1}\right)^{-1} . \tag{34c}$$

Here ν_t and Pe are given by equations (24b) and (26a) and χ_t is given by equation (11f) with $\nu \to 0$.

7. DISSIPATION

The dynamic equation for ϵ is given by (Canuto et al. 1994)

$$\frac{\partial\epsilon}{\partial t} + D_f(\epsilon) = c_1\epsilon K^{-1}P_b - c_2\epsilon^2K^{-1} + c_3\epsilon N , \tag{35a}$$

$$D_f(\epsilon) = -\frac{1}{2}\frac{\partial}{\partial z}\left[(\nu_t + \chi_t)\,\frac{\partial\epsilon}{\partial z}\right] , \tag{35b}$$

where P is the total production (in the case of pure buoyancy, $P_b = g\alpha J$) and $N \equiv (g\alpha\,|\,\beta\,|)^{1/2}$. The constants are $c_1 = 1.44$, $c_2 = 1.92$. The constant c_3 is nonzero only in the stably stratified case, $\nabla - \nabla_{\mathrm{ad}} < 0$ and zero otherwise. In the local limit, we have

$$\frac{\partial}{\partial z}\left[(\nu_t + \chi_t)\,\frac{\partial\epsilon}{\partial z}\right] \sim \nu_t l^{-2}\epsilon . \tag{35c}$$

Equating this with for example, the second term on the right-hand side gives

$$\epsilon \sim \frac{K^{3/2}}{l} , \tag{35d}$$

that is, equation (5c), which requires the specification of the mixing length l. It has recently been shown (Canuto & Dubovikov 1997d) that equation (35d), together with the other equations (19a)–(19d), may lead to divergent results, and thus it is preferable to use equation (35a).

8. DIFFUSION

Finally, we have to compute the nonlocal terms represented by the diffusion D_f in equations (19a)–(19d). The best model we can suggest is the one in which the third-order moments satisfy the dynamical equations which are obtained using the same procedure employed to obtain the second-order moments. We begin by recalling that

$$D_f(K) \equiv \frac{\partial}{\partial z}\left(\frac{1}{2}\,\overline{q^2w}\right) , \quad D_f\!\left(\frac{1}{2}\,\overline{\theta^2}\right) \equiv \frac{\partial}{\partial z}\left(\frac{1}{2}\,\overline{\theta^2w}\right) , \tag{36a}$$

$$D_f(J) \equiv \frac{\partial}{\partial z}\,(\overline{\theta w^2}) , \quad D_f\!\left(\frac{1}{2}\,\overline{w^2}\right) \equiv \frac{\partial}{\partial z}\left(\frac{1}{2}\,\overline{w^3}\right) . \tag{36b}$$

We have (Canuto 1992)

$$\left(\frac{\partial}{\partial t} + 2c\tau^{-1}\right)\overline{\theta w^2} = \beta\overline{w^3} - J\,\frac{\partial}{\partial z}\,\overline{w^2} + dg\alpha\overline{\theta^2w} - 2\overline{w^2}\,\frac{\partial J}{\partial z} , \tag{37a}$$

$$\left(\frac{\partial}{\partial t} + 2c\tau^{-1} + 2\tau_\theta^{-1}\right)\overline{w\theta^2} = 2\beta\overline{\theta w^2} - 2J\,\frac{\partial J}{\partial z} + eg\alpha\overline{\theta^3} - \overline{w^2}\,\frac{\partial\overline{\theta^2}}{\partial z} , \tag{37b}$$

$$\left(\frac{\partial}{\partial t} + 2c\tau^{-1}\right)\overline{w^3} = -3\overline{w^2}\,\frac{\partial}{\partial z}\,\overline{w^2} + 3eg\alpha\overline{\theta w^2} - 2\tau^{-1}\overline{q^2w} , \tag{37c}$$

$$\left(\frac{\partial}{\partial t} + 2c_*\tau^{-1}\right)\overline{wq^2} = -\left(2\overline{w^2}\,\frac{\partial}{\partial z}\,\overline{w^2} + \overline{w^2}\,\frac{\partial}{\partial z}\,\overline{q^2}\right) + eg\alpha(2\overline{\theta w^2} + \overline{q^2\theta}) , \tag{37d}$$

$$\left(\frac{\partial}{\partial t} + 2c\tau^{-1}\right)\overline{\theta q^2} = \beta\overline{q^2w} - \left(2\overline{w^2}\,\frac{\partial J}{\partial z} + J\,\frac{\partial}{\partial z}\,\overline{q^2}\right) + 2g\alpha\overline{w\theta^2} , \tag{37e}$$

$$\left(\frac{\partial}{\partial t} + 2c_{**}\tau^{-1}\right)\overline{\theta^3} = 3\beta\overline{\theta^2w} - 3J\,\frac{\partial\overline{\theta^2}}{\partial z} + \chi\,\frac{\partial^2}{\partial z^2}\,\overline{\theta^3} , \tag{37f}$$

where

$$c = 8 , \quad d = \tfrac{26}{15} , \quad e = \tfrac{4}{5} , \quad c_* = c + \tfrac{5}{3} , \quad c_{**} = c - 2 . \tag{37g}$$

8.1. *Down-Gradient Approximation*

The simplest model corresponds to retaining in equations (37a)–(37d) only the terms corresponding to the analog of equation (1). We obtain

$$\overline{\theta w^2} = -D_1\,\frac{\partial J}{\partial z} , \quad \overline{w\theta^2} = -D_2\,\frac{\partial\overline{\theta^2}}{\partial z} ,$$

$$\overline{w^3} = -\frac{3}{2}\,D_1\,\frac{\partial}{\partial z}\,\overline{w^2} , \quad \overline{wq^2} = -\frac{1}{2}\,\frac{c}{c_*}\,D_1\,\frac{\partial}{\partial z}\,\overline{q^2} , \tag{38a}$$

where the two turbulent diffusivities D values are defined as

$$D_1 \equiv c^{-1}\tau\overline{w^2}\ , \quad D_2 \equiv \tfrac{1}{2}c(c + \tau\tau_\theta^{-1})^{-1}D_1\ . \tag{38b}$$

It has been amply discussed in the literature (Moeng & Wyngaard 1986, 1988; Canuto et al. 1994) that equations (38a) and (38b) yield results that fail quite conspicuously to match LES data.

8.2. *Intermediate Solution*

In this model, we take two of the third-order moments as given by (see derivation in eqs. [50b] and [50c])

$$\overline{w\theta^2} = J\psi_1\ , \quad \psi_1 = S_w(\overline{\theta^2})^{1/2}\ ,$$
$$\overline{w^2\theta} = J\psi_2\ , \quad \psi_2 = S_w(\overline{w^2})^{1/2}\ , \tag{38c}$$

where S_w is the skewness of the velocity field

$$S_w = \frac{\overline{w^3}}{(\overline{w^2})^{3/2}}\ . \tag{38d}$$

Solving equations (37c)–(37e), we then obtain the other third-order moments:

$$s^{-1}\overline{w^3} = -A_1 - 2\overline{wq^2}\ ,$$
$$p^{-1}\overline{wq^2} = -A_4 + A_5\overline{w^3}\ , \tag{38e}$$

where the A_k values are second-order moments, specifically:

$$A_1 = 3\tau\overline{w^2}\,\frac{\partial}{\partial z}\,\overline{w^2}\ ,$$

$$A_2 = 2\overline{w^2}\,\frac{\partial}{\partial z}\,\overline{w^2} + \overline{w^2}\,\frac{\partial}{\partial z}\,\overline{q^2}\ ,$$

$$A_3 = 2\overline{w^2}\,\frac{\partial J}{\partial z} + J\,\frac{\partial}{\partial z}\,\overline{q^2}\ ,$$

$$2c_* A_4 = \tau A_2 + \frac{e}{2c}\,g\alpha\tau^2 A_3\ ,$$

$$c_* A_5 = eg\alpha\tau J(\overline{w^2})^{-1}\left[1 + \frac{1}{2c}\,g\alpha\tau\left(\frac{\overline{\theta^2}}{\overline{w^2}}\right)^{1/2}\right],$$

$$p^{-1} \equiv 1 - \frac{e}{4cc_*}\,g\alpha\beta\tau^2\ ,$$
$$s^{-1} = 2c - 3eg\alpha\tau J(\overline{w^2})^{-1}\ . \tag{38f}$$

The four third-order moments entering equations (36a) and (36b) are thus fully expressed in terms of second-order moments.

8.3. *Stationary*

The full solution of equations (37a)–(37f) in the stationary case and without radiative losses can also be worked out using methods of symbolic algebra (Canuto 1993). The results were successfully tested against LES data for the PBL (Canuto et al. 1994). In compact form, the result can be presented as follows. We first define a one-column vector $\boldsymbol{T}$:

$$D_f(K) = \frac{\partial}{\partial z}\,T_{11}\ , \quad D_f\left(\frac{1}{2}\,\overline{w^2}\right) = \frac{\partial}{\partial z}\,T_{21}\ , \tag{39a}$$

$$D_f(J) = \frac{\partial}{\partial z}\,T_{31}\ , \quad D_f\left(\frac{1}{2}\,\overline{\theta^2}\right) = \frac{\partial}{\partial z}\,T_{41}\ . \tag{39b}$$

Similarly, we define the one-column vector $\boldsymbol{S}$ such that

$$S_{11} \equiv K\ , \quad S_{21} \equiv \tfrac{1}{2}\overline{w^2}\ , \quad S_{31} \equiv J\ , \quad S_{41} \equiv \tfrac{1}{2}\overline{\theta^2}\ . \tag{39c}$$

The third-order moments are then related to the second-order moments by

$$T_{ij} = M_{ik}\,\frac{\partial}{\partial z}\,S_{kj}\ , \tag{39d}$$

where the matrix M_{ij} is given by

$$M_{11} = E_4\ , \quad M_{12} = E_2\ , \quad M_{13} = \tfrac{1}{2}g\alpha\tau E_1\ ,$$
$$M_{14} = (g\alpha\tau)^2 E_3\ , \tag{39e}$$

$$M_{21} = B_4\ , \quad M_{22} = B_2\ , \quad M_{23} = \tfrac{1}{2}g\alpha\tau B_1\ ,$$
$$M_{24} = (g\alpha\tau)^2 B_3\ , \tag{39f}$$

$$M_{31} = 2(g\alpha\tau)^{-1}A_4\ , \quad M_{32} = 2(g\alpha\tau)^{-1}A_2\ ,$$
$$M_{33} = A_1\ , \quad M_{34} = 2g\alpha\tau A_3\ , \tag{39g}$$

$$M_{41} = (g\alpha\tau)^{-2}C_4\ , \quad M_{42} = (g\alpha\tau)^{-2}C_2\ ,$$
$$M_{43} = \tfrac{1}{2}(g\alpha\tau)^{-1}C_1\ , \quad M_{44} = C_3\ . \tag{39h}$$

The functions A, B, C, and E (cm^2 s^{-1}) have the general form

$$A_k = A_{k_1}\tau\overline{w^2} + A_{k_2}\,g\alpha\tau^2 J\ . \tag{39i}$$

The A_{k_1}, A_{k_2}, etc. are given in Appendix B of Canuto (1993). This form of the third-order moments were shown to compare very well with the LES data (Canuto et al. 1994).

Depending on the specific problem, it may not however, be possible to assume the stationary case, in which case one has to evolve in time equations (37a)–(37f).

9. FULL NONLOCAL MODEL

In summary, the complete nonlocal model is composed of five differential equations for the five variables

$$K\ , \quad \overline{w^2}\ , \quad \tfrac{1}{2}\overline{\theta^2}\ , \quad \overline{w}\ , \quad \epsilon\ , \tag{40}$$

which represent the total turbulent kinetic energy, the turbulent pressure $p_t = \rho\overline{w^2}$, the potential energy, the convective flux and the rate of energy dissipation. The dynamic equations are given by equations (19a)–(19e) and (34a)–(34c). The diffusion terms are given by equations (36)–(37), ϵ is given by equations (35a)–(35b), ν_t and Pe are given by equatons (24b) and (26a), while χ_t is given by equation (11f) with $\nu \to 0$. The dissipation of temperature variance ϵ_θ is given by equation (20a). The constants γ_1 and β_5 are given by equation (23e).

10. STATIONARY AND LOCAL LIMIT

Almost without exceptions, in the past 40 years stellar structure calculations treated convection with a stationary, local model,

$$\frac{\partial}{\partial t} \to 0\ , \quad \frac{\partial}{\partial z} \to l^{-1}\ , \tag{41}$$

where l is a mixing length. Such approximation holds best in the efficient convective region but the lack of diffusion does not allow these models to incorporate the OV phenomenon. To this category belong the MLT model, the CM model (Canuto & Mazzitelli 1991), and CGM model (Canuto, Goldman, & Mazzitelli 1996). The dissipation ϵ is given by equation (5c). Since the problem is now algebraic, it can be solved analytically. The results are the following:

Convective flux (in units of $c_p \rho$):

$$F_c = \overline{w\theta} \equiv J \ , \quad J = \beta\chi\Phi \ , \tag{42a}$$

$$\Phi = \mathrm{Ko}^3 \ C\left(\frac{\tau_{p\theta}}{\tau}\right)^{3/2} S^{1/2} \ , \quad S \equiv g\alpha\beta l^4 \chi^{-2} \tag{42b}$$

$$C = \left(\frac{27}{\pi^4}\right)^{1/2}\left[1 + 2\beta_5 \frac{\tau_{pv}}{\tau} + 3(1-\gamma_1)\frac{\tau_\theta}{\tau}\right]^{3/2} . \tag{42c}$$

In the case of efficient convection, $\tau_{p\theta}/\tau$ and τ_θ/τ are constant, equation (34), and we derive

$$\Phi \sim S^{1/2} . \tag{43a}$$

In the case of inefficient convection, we first obtain, using equation (34),

$$\tau_{p\theta}/\tau = (8\pi^2)^{-1} c_\epsilon^{2/3} (S\Phi)^{1/3} \ , \quad \tau_\theta/\tau = 2(7\pi^2)^{-1} c_\epsilon^{2/3} (S\Phi)^{1/3} \ , \tag{43b}$$

which implies, for $\mathrm{Ko} = 5/3$,

$$\Phi = 2 \times 10^{-5} S^2 \ , \tag{43c}$$

which is intermediate between the CM and the CGM models.

Turbulent kinetic energy, K:

$$K = \frac{3}{2}\pi^{-2/3}\ \mathrm{Ko}\ (S\Phi)^{2/3}\left(\frac{\chi}{l}\right)^2 ; \tag{44a}$$

Vertical turbulent kinetic energy:

$$\frac{1}{2}\overline{w^2} = \frac{1}{3}K\left(1 + 2\beta_5 \frac{\tau_{pv}}{\tau}\right); \tag{44b}$$

Temperature variance, $(1/2)\overline{\theta^2}$*:*

$$\frac{1}{2}\overline{\theta^2} = c_\epsilon^{-2/3}\left(\frac{\tau_\theta}{\tau}\right)\Phi(\beta l)^2 (S\Phi)^{-1/3} ; \tag{44c}$$

Potential to kinetic energy ratio:

$$\frac{1}{2}\frac{g\alpha\overline{\theta^2}}{\beta K} = \frac{\tau_\theta}{\tau} . \tag{44d}$$

The basic predictions of the CM and CGM models, namely, (1) MLT underestimates Φ in the effective convective region, (2) MLT overestimates Φ in the low convective efficiency region, (3) the mixing length $l = z$, have all been confirmed either by direct numerical simulation LES (Demarque, Guenther, & Kim 1997) or via astrophysical tests (red giants, Stothers & Chin 1995; helioseismology, Basu & Antia 1994; Baturin & Miranova 1995; Antia & Basu 1997; white dwarfs, Althaus & Benvenuto 1996; low-mass stars, D'Antona & Mazzitelli 1996; stellar atmospheres Kupka 1996; $l = z$ rule, Stothers & Chin 1997).

11. NONLOCAL MODELS OF GOUGH AND XIONG

Xiong (1986) and Xiong et al. (1997) have suggested and worked out the consequences of a nonlocal model that employs the four differential equations (19a)–(19d) under the following approximations:

$$(\tau_{p\theta}, \tau_\theta) \sim \tau \ , \quad D_f\text{: down gradient} . \tag{45a}$$

The first approximation is valid in the very efficient regime, while in the OV region it is no longer valid since equations (34b) and (34c) show that

$$\tau_{p\theta}, \tau_\theta \ll \tau . \tag{45b}$$

An overestimate of the timescales may lead to an overestimate of the OV extent. As for ϵ, instead of equations (35a) and (35b), the model adopts a local expression, equation (5c). As recently shown (Canuto & Dubovikov 1997d), this can give rise to divergent results. The down-gradient model for the D_f values has already been discussed.

Gough (1976) has suggested a nonlocal model that adopts only two of the four equations (19a)–(19d), namely, the ones corresponding to $\overline{w^2}$ and J. The diffusion was treated with the down gradient form equations (38a)–(38f) and ϵ was treated with the local model given by equation (5c). The model equations can be written as

$$\frac{\partial^2}{\partial \xi^2} X = X - X^l \ , \quad X \equiv (J, p_t) . \tag{46a}$$

The superscript l means that one uses the local expressions, e.g., equations (42a)–(42*c*) and (44b); the variable ξ is defined as z/l, where l is the mixing length, $l = \alpha H_p$.

Consider first equation (19d). If we employ the down-gradient approximation, equation (38c), simple steps lead to (in the stationary case)

$$p_t'' + Ap_t' = p_t - p_t^l \ , \tag{46b}$$

where $(\partial/\partial\xi = {}')$

$$A = (\ln D_1)' \ , \quad \xi = z/\Lambda \ ,$$

$$\Lambda^2 = \tfrac{3}{4} D_1 \tau_{pv} \ , \quad p_t^l = \tfrac{2}{3} K(1 + 2\beta_5 \tau_{pv}\tau^{-1}) \ , \tag{46c}$$

and we have used $g\alpha J = \epsilon$ and equation (44b).

Next, consider the equation for J, equation (19c). Using the same procedure, we derive

$$J'' + AJ' = J - J_l \ , \tag{47a}$$

with

$$\Lambda^2 = \tfrac{1}{2} D_1 \tau_{p\theta} \ , \quad J_l \equiv (\beta\overline{w^2} + \tfrac{2}{3} g\alpha\overline{\theta^2})\tau_{p\theta} . \tag{47b}$$

These equations are valid for both efficient and inefficient convection.

Comparing equations (46b) and (47a) with equation (46a) we conclude that we are unable to reproduce Gough's model. To do so, we must neglect A and assume the same Λ in both p_t and J equations, while they are different since the first depends on τ_{pv} while the other depends on $\tau_{p\theta}$: the former does not depend on the Peclet number while the latter does. The neglect of A may be interpreted as follows. Consider the velocity field skewness

$$S_w = \frac{\overline{w^3}}{(\overline{w^2})^{3/2}} . \tag{48a}$$

Using equation (38c) and the fact that $\overline{w^2} \sim K$ and that $D_1 \sim lK^{1/2}$, we derive

$$S_w \sim -K^{-1}\frac{\partial K}{\partial \xi} \sim A . \tag{48b}$$

For a constant l, $A = 0$ is thus equivalent to $S_w = 0$. However, $S_w = 0$ is the property of a Gaussian field while turbulence is highly non-Gaussian; in fact, a turbulent field's most distinguishing feature is that of having a finite skewness. For the case of convective boundary layer, S_w is given in Figure 1 of Moeng & Rotunno (1990), whereas for the case of Benard-Rayleigh convection, see Kerr (1996).

Gough's model has recently been applied to helioseismology by Balmforth (1992) and Houdek (1997).

12. UP AND DOWNDRAFTS: CENTRAL ROLE OF SKEWNESS

It is known from the study of the convective boundary layer (Wyngaard 1987; Moeng & Rotunno 1990; Randall, Shao, & Moeng 1992) that a convective layer exhibits regions of updrafts and downdrafts. Here we shall discuss a simple model to highlight the central role of the skewness equation (48a) in determining the basic features of the up-down drafts. We shall write for the mean vertical velocity and temperature

$$\bar{w} = \sigma w_u + (1-\sigma) w_d \,, \tag{49a}$$

$$\bar{\theta} = \sigma \theta_u + (1-\sigma)\theta_d \,, \tag{49b}$$

meaning that the up (down) drafts have velocities $w_u(w_d)$ and occupy fractional areas σ and $(1-\sigma)$, respectively. We have

$$\begin{aligned} \overline{w^2} &= (w_u - \bar{w})^2\sigma + (w_d - \bar{w})^2(1-\sigma) \,, \\ \overline{\theta^2} &= (\theta_u - \bar{\theta})^2\sigma + (\theta_d - \bar{\theta})^2(1-\sigma) \,, \\ \overline{w\theta} &= (\theta_u - \bar{\theta})(w_u - \bar{w})\sigma + (w_d - \bar{w})(\theta_d - \bar{\theta})(1-\sigma) \,, \end{aligned} \tag{49c}$$

or, more explicitly,

$$\begin{aligned} \overline{w^2} &= \sigma(1-\sigma)(w_u - w_d)^2 \,, \\ \overline{\theta^2} &= \sigma(1-\sigma)(\theta_u - \theta_d)^2 \,, \\ \overline{w\theta} &= \sigma(1-\sigma)(w_u - w_d)(\theta_u - \theta_d) \,. \end{aligned} \tag{49d}$$

For the third-order moments, we have

$$\begin{aligned} \overline{w^3} &= \sigma(1-\sigma)(1-2\sigma)(w_u - w_d)^3 \,, \\ \overline{w^2\theta} &= \sigma(1-\sigma)(1-2\sigma)(w_u - w_d)^2(\theta_u - \theta_d) \,, \\ \overline{w\theta^2} &= \sigma(1-\sigma)(1-2\sigma)(\theta_u - \theta_d)^2(w_u - w_d) \,. \end{aligned} \tag{50a}$$

Combining the above relations, we further obtain

$$\overline{w\theta^2} = \psi_1 \overline{w\theta} \,, \quad \psi_1 = S_w(\overline{\theta^2})^{1/2} \,, \tag{50b}$$

$$\overline{w^2\theta} = \psi_2 \overline{w\theta} \,, \quad \psi_2 = S_w(\overline{w^2})^{1/2} \,, \tag{50c}$$

which relate two third-order moments to the second-order moment $\overline{w\theta}$ provided one knows the skewness S_w which is related to the filling factor σ or by the relation

$$\begin{aligned} \sigma &= \tfrac{1}{2}[1 - S_w(4 + S_w^2)^{-1/2}] \,, \\ S_w &= (1-2\sigma)[\sigma(1-\sigma)]^{-1/2} \,. \end{aligned} \tag{50d}$$

We further note that with $\bar{w} = 0$, we have

$$\begin{aligned} w_u &= \left(\frac{1-\sigma}{\sigma}\right)^{1/2} (\overline{w^2})^{1/2} \,, \\ w_d &= -\left(\frac{\sigma}{1-\sigma}\right)^{1/2} (\overline{w^2})^{1/2} \,, \\ \theta_u &= \bar{\theta} + \left(\frac{1-\sigma}{\sigma}\right)^{1/2} (\overline{\theta^2})^{1/2} \,, \\ \theta_d &= \bar{\theta} - \left(\frac{\sigma}{1-\sigma}\right)^{1/2} (\overline{\theta^2})^{1/2} \,. \end{aligned} \tag{50f}$$

The key ingredient is σ, the filling factor, and thus the knowledge of the skewness S_w is instrumental to determine the basic features of the "thermals." Numerical simulations (Stein & Nordlund 1989; Cattaneo et al. 1991; Spruit 1997) have shown several facts, among which the most salient are the following:

1. There exist fast, concentrated downflows in the midst of slow and broad upflows;
2. The convective flux carried by the downflows is larger than that carried by the upflows (in the middle of the cell, the ratio is about 2);
3. The flux of turbulent kinetic energy is negative and carried mostly by the downflows;
4. The " convected flux", that is,

$$F(\text{convected}) \equiv \tfrac{1}{2}\rho\overline{q^2 w} + c_p \rho\overline{w\theta} \equiv F_{\kappa e} + F_{\text{conv}} \,, \tag{51a}$$

in the downflows is almost zero because the two fluxes cancel each other out; and

5. The only remaining contribution to F(convected) comes from the disordered upflow where $F_{\kappa\epsilon} < F_c$.

Property (1) implies that the filling factor for upflows (σ) is *larger* than the filling factor for downflows $(1-\sigma)$. Thus, $\sigma > \frac{1}{2}$, $|w_d| > |w_u|$, and $S_w < 0$.

Regarding property (2), we use equation (49d) to write the enthalpy (convective) flux as the sum of up and down fluxes. Use of equation (50f) then gives (in units of $c_p\,\rho$)

$$\begin{aligned} F_c(\text{up}) &= (1-\sigma)(\overline{\theta^2})^{1/2}(\overline{w^2})^{1/2} \,, \\ F_c(\text{down}) &= \sigma(\overline{\theta^2})^{1/2}(\overline{w^2})^{1/2} \,. \end{aligned} \tag{51b}$$

If $\sigma > \frac{1}{2}$, property (2) is also satisfied.

As for property (3), when $S_w < 0$, $F_{\kappa e} < 0$.

The relevance of the above simplified model is that it provides a way to relate and cross-check the predictions of DNS/LES with those of the present model.

13. ROLE OF COMPRESSIBILITY

One may question the Boussinesq approximation since the rather large extent of the CZ in stars (in the Sun is about 30% of the radius) makes density variations quite substantial and thus arguably only poorly described by the Boussinesq model. In this context, we would like to offer the following observations: (1) by far the largest fraction of the CZ is so nearly adiabatic that a detailed theory of convection is actually hardly needed since $\nabla = \nabla_{\text{ad}}$ is an excellent approximation. A reliable theory of convection is needed in the much smaller region where convection becomes inefficient, which of course comprises the OV region. Observational data tell us that in massive stars OV $< 0.2H_p$, while in the Sun OV $< 0.05H_p$. These facts help the Boussinesq approximation more than in the case of Earth's planetary convective layer (PBL) whose extent of $\sim$1 km is small compared with $H_p \sim 8$ km, a fact considered more than sufficient justification for the Boussinesq approximation. Needless to say, radiative effects must nonetheless be accounted for. (2) A recent model of compressible convection (Canuto 1997a, 1997b) shows that it will probably lower the extent of OV which the present model predicts to be already within the upper limit set by the data. (3) The local, Boussinesq, one-eddy MLT model has been the standard tool for more than 40 years and only recently improvements are being included in stellar codes. The path toward a fully compressible model must go through intermediate

TABLE 1

Type of Model	Number of Equations[a]
Boussinesq type:	
Local, MLT	1A
Semilocal, CM, CGM	1A
Nonlocal, Gough 1976	3DE
Nonlocal, Xion et al. 1997	4DE
Nonlocal, Present model	5DE
Nonlocal, Compressible	18DE

[a] A = algebraic; DE = differential equations.

models like the present one since the number of equations involved increases quite substantially, as Table 1 shows.

14. CONCLUSIONS

Within the Boussinesq approximation, the present model represents state of the art in turbulence modeling. The assessment is based on three facts: (1) absence of adjustable parameters, (2) testing of the model on more than 80 statistics from laboratory, DNS and LES data, and (3) astrophysical tests of the local limit of the theory, specifically: red giants, (Stothers & Chin 1991, 1995), helioseismology (Basu & Antia 1994; Baturin & Miranova 1995; Antia & Basu 1997; Canuto & Christensen-Dalsgaard 1998), white dwarfs (Althaus & Benvenuto 1996), low mass stars (D'Antona & Mazzitelli 1996), stellar atmospheres (Kupka 1996), $l = z$ rule, (Stothers & Chin 1997).

It is hoped that the model will now be applied to quantify the OV phenomenon in the Sun as well as in massive stars where a reliable theoretical determination is still lacking while observational data have already provided quite stringent limits.

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REFERENCES

Althaus, L. G., & Benvenuto, O. G. 1996, MNRAS, 278, L33
Andersen, J., Nordstom, B., & Clausen, J. V. 1990, ApJ, 363, L33
Antia, H. M., & Basu, S. 1997, in Solar Convection, Oscillations and their Relationship, SCORE 1996, ed. F. P. Pijpers, J. Christensen-Dalsgaard, & C. S. Rosenthal (Dordrecht: Kluwer)
Balmforth, N. J. 1992, MNRAS, 255, 603
Basu, S. 1997, MNRAS, 288, 572
Basu, S., & Antia, H. M. 1994, J. Astrophys. Astron., 15, 143
Batchelor, G. K. 1971, The Theory of Homogeneous Turbulence (Cambridge: Cambridge Univ. Press)
Baturin, V. A., & Miranova, I. V. 1995, Astron. Rep., 39, 105
Canuto, V. M. 1992, ApJ, 392, 218
———. 1993, ApJ, 416, 331
———. 1997a, ApJ, 478, 322
———. 1997b, ApJ, 482, 827
Canuto, V. M., & Christensen-Dalsgaard, J. 1998, Ann. Rev. Fluid Mech., in press
Canuto, V. M., & Dubovikov, M. 1996a, Phys. Fluids, 8, 572 (Paper I)
———. 1996b, Phys. Fluids, 8, 587 (Paper II)
———. 1996c, Phys. Fluids, 8, 599 (Paper III)
———. 1997a, Phys. Fluids, 9, 2118 (Paper IV)
———. 1997b, Phys. Fluids, 9, 2132 (Paper V)
———. 1997c, Phys. Fluids, 9, 2141 (Paper VI)
———. 1997d, ApJ, 484, L161
Canuto, V. M., Goldman, I., & Chasnov, J. 1988, Phys. Fluids, 30, 3391
Canuto, V. M., Goldman, I., & Mazzitelli, I. 1996, ApJ, 473, 550
Canuto, V. M., & Mazzitelli, I. 1991, ApJ, 370, 295
Canuto, V. M., Minotti, F. O., Ronchi, C., & Zeman, O. 1994, J. Atmos. Sci., 51, 12, 1605
Cattaneo, F., Brummel, N., Toomre, J., Malagoli, A., & Hurlburt, N. 1991, ApJ, 370, 282
Chan, K. W., & Sofia, S., 1989, ApJ, 336, 1022
———. 1996, ApJ, 466, 372
Chasnov, J. R. 1991, Phys. Fluids, A3, 188
D'Antona, F., & Mazzitelli, I. 1996, ApJ, 470, 1093
DeDominicis, C., & Martin, P. C. 1979, Phys. Rev. A, 19, 419
Demarque, P., Guenther, D. B., & Kim, Y. C. 1997, ApJ, 474, 790
Foster, D., Nelson, D. R., & Stephen, M. J. 1977, Phys. Rev. A, 16, 732
Gough, D. 1976, in IAU Colloq. 38, Problems of Stellar Convection, ed. E. Spiegel & J. P. Zahn (New York: Springer), 15
Herring, J. R., & Kraichman, R. H. 1971, in Statistical Models and Turbulence, ed. M. Rosenblatt & C. van Atta (New York: Springer), 147–194
Houdek, G. 1997, Ph.D. thesis, Inst. Astron., Vienna
Kerr, R. M. 1996, J. Fluid Mech., 310, 139
Kozhurina-Platais, V., Demarque, P., Platais, I., & Orosz, J. A. 1997, AJ, 113, 1045
Kraichnan, R. H. 1970, J. Fluid Mech., 41, 189
Kupka, F. 1996, in Model Atmospheres and Spectrum Synthesis, ed. S. J. Adelman, F. Kupka, & W. W. Weiss (San Francisco: ASP), vol. 108, 73
Landau, L. D., & Lifschitz, E. M. 1970, Fluid Mechanics (New York: Pergamon Press)
Leith, C. 1971, J. Atmos. Sci., 28, 145
Lesieur, M. 1991, Turbulence in Fluids (Dordrecht: Kluwer)
Leslie, D. C. 1973, Developments in the Theory of Turbulence (Oxford: Clarendon Press)
McComb, W. D. 1990, The Physics of Fluid Turbulence, Oxford Science Publ.
Moeng, C. H., & Rotunno, R. 1990, J. Atmos. Sci., 47, 1149
Moeng, C. H., & Wyngaard, J. C. 1986, J. Atm. Sci., 43, 2499
———. 1988, J. Atm. Sci., 45, 3573
Monin, A. S., & Yaglom, A. M. 1975, Statistical Fluid Mechanics (Cambridge, MA: MIT)
Nordstrom, B., Andersen, J., & Andersen, M. I. 1997, A&A, 322, 460
Praskovsky, A., & Oncley, S. 1994, Phys. Fluids, 6, 2886
Priestly, C. H. B., & Swinbank, W. C. 1947, Proc. R. Soc. London A, 189, 543
Randall, D. A., Shao, Q., & Moeng, C. H. 1992, J. Atmos. Sci., 49, 1903
Roxburgh, I. W., & Vorontsov, S. V. 1994, MNRAS, 268, 889
Shaller, G., Schaerer, D., Meynet, G., & Maeder, A. 1992, A&AS, 96, 260
Singh, H. P., Roxburgh, I. W., & Chan, K. L. 1995, A&A, 295, 703
Spruit, H. C. 1997, Mem. Soc. Astron. Italian, in press
Stein, R., & Nordlund, A. 1989, ApJ, 342, L95
Stothers, R. B., & Chin, C. 1991, ApJ, 381, L67
———. 1995, ApJ, 440, 297
———. 1997, ApJ, 478, L103
Wilson, K. G., & Kogut, J. B. 1974, Phys. Rep., 12, 77
Wyld, H. W. 1961, Ann. Phys., 14, 143
Wyngaard, J. C. 1987, J. Atmos. Sci., 44, 1083
Xiong, D. R. 1986, A&A, 167, 239
Xiong, D. R., Cheng, Q. L., & Deng, L. 1997, ApJS, 108, 529